

A paradigm for the characterization of artifacts in tomography

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Abstract

We present a paradigm for characterization of artifacts in limited data tomography problems. In particular, we use this paradigm to characterize artifacts that are generated in reconstructions from limited angle data with generalized Radon transforms and general filtered backprojection type operators. In order to find when visible singularities are imaged, we calculate the symbol of our reconstruction operator as a pseudodifferential operator.

Keywords: Computed Tomography, Lambda Tomography, Limited Angle Tomography, Radon transforms, Microlocal Analysis, Fourier integral operators.

1 Introduction

In this article, we consider the generalized Radon transform integrating over lines in the plane. Let $s \in \mathbb{R}$, $\phi \in [0, 2\pi]$ and $\theta(\phi) = (\cos(\phi), \sin(\phi))$ be the unit vector in S^1 in direction ϕ and $\theta^\perp(\phi) = (-\sin(\phi), \cos(\phi))$, then $\theta^\perp(\phi)$ is perpendicular to $\theta(\phi)$. Let $\Xi = [0, 2\pi] \times \mathbb{R}$, then for each $(\phi, s) \in \Xi$, $L(\phi, s) = \{x \in \mathbb{R}^2 : x \cdot \theta(\phi) = s\}$ is the line containing $s\theta(\phi)$ and normal to $\theta(\phi)$. We let $\mu(\phi, x)$ be a smooth function on $\mathbb{R} \times \mathbb{R}^2$ that is 2π -periodic in ϕ . Then, we define the generalized Radon transform

$$R_\mu f(\phi, s) = \int_{x \in L(\phi, s)} f(x) \mu(\phi, x) \, dx \quad (1)$$

where dx denotes the arc length measure on the line. This transform integrates functions along lines.

We define the corresponding dual transform (or the backprojection operator) for $g \in \mathcal{S}(S^1 \times \mathbb{R})$ as

$$R_\mu^* g(x) = \int_0^{2\pi} g(\phi, x \cdot \theta(\phi)) \mu(\phi, x) \, d\phi, \quad (2)$$

which is the integral of g over all lines through x (since for each $\theta(\phi)$, $x \in L(\phi, x \cdot \theta(\phi))$). Note that authors, including Beylkin and others, use the weight $1/\mu$ for a different weighted dual operator. These transforms are both defined and weakly continuous for classes of distributions [5]. Many inversion formulas have been proven for the classical Radon transform ($\mu \equiv 1$) [12], and invertibility of R_μ has been well studied (e.g., [1, 14, 17]). Among the

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most prominent reconstruction formulas are those of filtered backprojection type [1, 12, 10] which have the following form

$$Bg = R_\mu^* P g, \quad \text{where } g = R_\mu f, \quad (3)$$

and P is a pseudodifferential operator that “filters” the data $g = R_\mu f$. For example, in case of the classical Radon transform and $P = (1/4\pi)\sqrt{-\partial^2/\partial s^2}$, (3) is an exact reconstruction formula and the basis for the filtered backprojection (FBP) algorithm [12]. Another prominent example is the so-called Lambda reconstruction formula which uses the filter $P = (1/4\pi)(-\partial^2/\partial s^2)$ in (3).

In this paper, we consider the problem of reconstructing f from incomplete data. More precisely, we assume that $R_\mu f(\phi, s)$ is known only for a limited angular range $\phi \in (a, b)$ (note that for $b - a \geq \pi$, every line can be parameterized by $\phi \in (a, b)$ although for general μ , the measure might be different on the line $L(\phi, s)$ and $L(\phi + \pi, -s)$ — $\mu(\phi, x)$ might not equal $\mu(\phi + \pi, x)$ for all (ϕ, x)). Thus, we deal with the restricted (or limited angle) generalized Radon transform which we define as

$$R_{\mu, (a,b)} f(\phi, s) = \chi_{(a,b) \times \mathbb{R}}(\phi, s) \cdot R_\mu f(\phi, s), \quad (4)$$

where $\chi_{(a,b) \times \mathbb{R}}$ denotes the characteristic function of the data space $(a, b) \times \mathbb{R}$ with the limited angular range (a, b) with $b - a < \pi$ (or $b - a < 2\pi$ if μ is not symmetric). Such limited angle problems arise in many practical situations and the filtered backprojection type reconstruction of the form (3) is still one of the preferred reconstruction methods [16]. It is well known that in this situation only visible singularities can be reconstructed reliably [18] and that the reconstruction problem is severely ill-posed [11, 12]. Moreover, it has been shown in [2, 8] that additional artifacts can be generated. In [2, 8], the authors consider the limited angle FBP and Lambda reconstructions for the classical limited angle tomography data $g_{(a,b)} = R_{(a,b)} f$ (i.e. $\mu \equiv 1$ and $P = \sqrt{-d^2/ds^2}$ for FBP and $P = -d^2/ds^2$ for Lambda) and derive a precise geometric characterization of artifacts. In particular, they show artifacts are generated along straight lines that are tangent to singularities of f whose directions correspond to the ends of the angular range. In [13], L. Nguyen characterized the strength of these artifacts. In [2, 8] the authors prove that a simple artifact reduction strategy smooths the artifacts. The same reduction strategy is proposed in [9] for R_μ and the Lambda and FBP filter, and the symbols are calculated for those specific operators for limited angle and ROI data.

The methods of [2, 8, 13] do not directly apply to the limited angle problem for the generalized Radon transform with reconstruction operators (3) (with P being an arbitrary pseudodifferential operator). This is mainly due to the fact that their proofs rely on explicit expressions of the operators as singular pseudodifferential operators.

In this paper, we study the application of the reconstruction operators (3) to the limited angle data for an arbitrary μ which is smooth and nowhere zero. Using the framework of microlocal analysis and the calculus of Fourier integral operators, we prove a qualitative characterization of artifacts and provide an artifact reduction strategy. Our proofs use the technique that was originally developed in [3] to characterize artifacts in photoacoustic tomography and sonar. In particular, we show that the visible and added singularities are contained in the same set as was obtained for specific cases in [2, 8]. We show that the artifact reduction strategy in [2, 8, 9] applies for general filters P (Theorem 5.1) and we show for some choices of P that most of the visible singularities are recovered by the artifact reduced reconstruction operator (Corollary 5.2).

The rest of the article is organized as follows. Basic definitions and notations are given in Section 2. In Section 3 we present a general paradigm to characterize added singularities in

limited view tomography. The characterization of limited angle artifacts for the generalized Radon transform is proven in Section 4, and the artifact reduction strategy and symbol calculations are presented in Section 5.

2 Notation

Let Ω be an open set. We denote the set of C^∞ functions with domain Ω , by $\mathcal{E}(\Omega)$ and the set of C^∞ functions of compact support in Ω by $\mathcal{D}(\Omega)$. Distributions are continuous linear functionals on these function spaces. The dual space to $\mathcal{D}(\Omega)$ is denoted $\mathcal{D}'(\Omega)$ and the dual space to $\mathcal{E}(\Omega)$ is denoted $\mathcal{E}'(\Omega)$. In fact, $\mathcal{E}'(\Omega)$ is the set of distributions of compact support in Ω . For more information about these spaces we refer to [19].

We will use the framework of microlocal analysis for our characterizations. Here, the notion of a wavefront set of a distribution $f \in \mathcal{D}'(\Omega)$ is central. It simultaneously describes the locations and directions of singularities of f . That is, f has a singularity at $x_0 \in \Omega$ in direction $\xi_0 \in \mathbb{R}^n \setminus \mathbf{0}$ if for any cutoff function φ at x_0 , the Fourier transform $\mathcal{F}(\varphi f)$ does not decay rapidly in any open conic neighborhood of the ray $\{t\xi_0 : t > 0\}$. Then, the *wavefront set* of $f \in \mathcal{D}'(\Omega)$, $\text{WF}(f)$, is defined as the set of all tuples (x_0, ξ_0) such that f is singular at x_0 in direction ξ_0 . As defined, $\text{WF}(f)$, is a closed subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \mathbf{0})$ that is conic in the second variable. However, in what follows, we will view the wavefront set as a subset of a cotangent bundle so it will be invariantly defined on manifolds [20].

We recall that, for a manifold Ξ and $y \in \Xi$, the cotangent space of Ξ at y , $T_y^*(\Xi)$ is the vector space of all first order differentials (the dual space to the tangent space $T_y(\Xi)$), and the cotangent bundle $T^*(\Xi)$ is the vector bundle with fiber $T_y^*(\Xi)$ above $y \in \Xi$. That is $T^*(\Xi) = \{(y, \eta) : y \in \Xi, \eta \in T_y^*(\Xi)\}$. The differentials $\mathbf{d}\mathbf{x}_1, \mathbf{d}\mathbf{x}_2, \dots$, and $\mathbf{d}\mathbf{x}_n$ are a basis of $T_x^*(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$. For $\xi \in \mathbb{R}^n$, we will use the notation

$$\xi \mathbf{d}\mathbf{x} = \xi_1 \mathbf{d}\mathbf{x}_1 + \xi_2 \mathbf{d}\mathbf{x}_2 + \dots + \xi_n \mathbf{d}\mathbf{x}_n \in T_x^*(\mathbb{R}^n).$$

If $\phi \in \mathbb{R}$ then $\mathbf{d}\phi$ will be the differential with respect to ϕ , and differentials $\mathbf{d}\mathbf{r}$ and $\mathbf{d}\mathbf{s}$ are defined analogously.

Let X and Y be manifolds, and $C \subset T^*(Y) \times T^*(X)$, then

$$C^t = \{(x, \xi; y, \eta) : (y, \eta; x, \xi) \in C\}. \quad (5)$$

If $D \subset T^*(X)$, we define

$$C \circ D = \{(y, \eta) \in T^*(Y) : \exists (x, \xi) \in D : (y, \eta; x, \xi) \in C\}. \quad (6)$$

Fourier integral operators (FIO) are linear operators on distribution spaces that precisely transform wavefront sets. They are defined in [6, 20] in terms of amplitudes and phase functions. If X and Ξ are manifolds and $\mathcal{F} : \mathcal{D}'(X) \rightarrow \mathcal{D}'(\Xi)$ is a FIO, then associated to \mathcal{F} is the *canonical relation* $C \subset T^*(\Xi) \times T^*(X)$. Then the Hörmander-Sato Lemma (e.g., [20, Th. 5.4, p. 461]) asserts for $f \in \mathcal{E}'(X)$ that

$$\text{WF}(\mathcal{F}f) \subset C \circ \text{WF}(f). \quad (7)$$

3 The paradigm

In this section, we will present a methodology that can be used to prove characterizations of limited view artifacts for a number of tomography problems. In the next section, we will apply them to R_μ .

This methodology was originally developed in [3] to understand visible and added singularities in limited data photoacoustic tomography and sonar. Denote the forward operator by $\mathcal{M} : \mathcal{E}'(\Omega) \rightarrow \mathcal{E}'(\Xi)$ and assume \mathcal{M} is a FIO. The *object space* Ω is a region to be imaged and the *data space* Ξ is a space that parameterizes the data. A *limited data problem* for \mathcal{M} will be a specification of an open subset $A \subset \Xi$ on which data are given, and in this case, the limited data operator can be written

$$\mathcal{M}_A f = \chi_A \mathcal{M} f, \quad (8)$$

where χ_A is the characteristic function of A and the product just restricts the data to the set A . In the cases we consider, the reconstruction operator is of the form

$$\mathcal{M}^* P \mathcal{M}_A, \quad (9)$$

where \mathcal{M}^* is an appropriate dual or backprojection operator to \mathcal{M} , and this models our reconstruction operator (3).

Our next theorem tells what multiplication by χ_A does to the wavefront set. It is a special case of Theorem 8.2.10 in [7].

Theorem 3.1. *Let u be a distribution and let A be a closed subset of Ξ with nontrivial interior. If the non-cancellation condition*

$$\forall (y, \xi) \in T^*(\Xi), \quad (y, \xi) \in \text{WF}(u) \text{ iff } (y, -\xi) \notin \text{WF}(\chi_A) \quad (10)$$

holds, then the product $\chi_A u$ can be defined as a distribution. In this case, we have

$$\text{WF}(\chi_A u) \subset \mathcal{Q}(A, \text{WF}(u)), \quad (11)$$

where for $A \subset \Xi$ and $W \subset T^(\Xi)$*

$$\begin{aligned} \mathcal{Q}(A, W) := & \{ (y, \xi + \eta) : y \in A, [(y, \xi) \in W \text{ or } \xi = 0] \\ & \text{and } [(y, \eta) \in \text{WF}(\chi_A) \text{ or } \eta = 0] \}. \end{aligned} \quad (12)$$

Note that the condition “ $y \in A$ ” is not in (12) in Hörmander’s theorem, but we include this because χ_A is zero (and so smooth) off of A . Also, note that the case $\xi = \eta = 0$ in the definition of \mathcal{Q} is not allowed since the wavefront set does not include zero vectors.

Our paradigm for proving characterizations for visible and added artifacts is given by the following procedure, cf. [3]:

- (a) Confirm the forward operator \mathcal{M} is a FIO and calculate its canonical relation, C .
- (b) Choose the limited data set $A \subset \Xi$ and calculate $\text{WF}(\chi_A)$.
- (c) Make sure the non-cancellation condition (10) holds for χ_A and $\mathcal{M}f$. This can be done in general by making sure it holds for $(y, \eta) \in C \circ (T^*(\Omega) \setminus \mathbf{0})$.
- (d) Calculate $\mathcal{Q}(A, C \circ \text{WF}(f))$.
- (e) Calculate $C^t \circ \mathcal{Q}(A, C \circ \text{WF}(f))$ to find possible visible singularities and added artifacts using [3, Lemma 3.2]:

$$\text{WF}(\mathcal{M}^* P \mathcal{M}_A f) \subset C^t \circ \mathcal{Q}(A, C \circ \text{WF}(f)). \quad (13)$$

4 Characterization of artifacts

The first proposition provides the microlocal properties of R_μ and R_μ^* .

Proposition 4.1. *If μ is nowhere zero, then the generalized Radon transform R_μ is an elliptic Fourier integral operator associated to the canonical relation*

$$C = \{((\phi, s), \alpha [-x \cdot \theta^\perp(\phi) d\phi + ds] ; x, \alpha \theta(\phi) dx) : (\phi, s) \in [0, 2\pi] \times \mathbb{R}, \alpha \neq 0, x \in L(\phi, s)\}. \quad (14)$$

The dual operator R_μ^* is an elliptic Fourier integral operator associated to the canonical relation C^t defined in (5).

Let $\Pi_R : C \rightarrow T^*(\mathbb{R}^2)$ and $\Pi_L : C \rightarrow T^*(\Xi)$ be the natural projections. Then Π_L is an injective immersion and Π_R is a two-to-one immersion. Let $(x, \xi \mathbf{dx}) \in T^*(\mathbb{R}^2) \setminus \mathbf{0}$. Let $\phi_0 = \phi_0(\xi)$ be the unique angle in $[0, 2\pi)$ with $\xi = \|\xi\| \theta(\phi_0)$ and let $\phi_1 = \phi_1(\xi)$ be the unique angle in $[0, 2\pi)$ with $\xi = -\|\xi\| \theta(\phi_0)$. Define

$$\begin{aligned} \lambda_0(x, \xi) &= (\phi_0(\xi), x \cdot \theta(\phi_0(\xi)), \|\xi\| [-x \cdot \theta^\perp(\phi_0(\xi)) d\phi + ds]) \\ \lambda_1(x, \xi) &= (\phi_1(\xi), x \cdot \theta(\phi_1(\xi)), -\|\xi\| [-x \cdot \theta^\perp(\phi_1(\xi)) d\phi + ds]). \end{aligned} \quad (15)$$

The two preimages of $(x, \xi \mathbf{dx})$ under Π_R are $(\lambda_0(x, \xi); x, \xi \mathbf{dx})$ and $(\lambda_1(x, \xi); x, \xi \mathbf{dx})$. Therefore,

$$\begin{aligned} C \circ \{(x, \xi \mathbf{dx})\} &= \{\lambda_0(x, \xi), \lambda_1(x, \xi)\} \\ C^t \circ \{\lambda_0(x, \xi \mathbf{dx})\} &= C^t \circ \{\lambda_1(x, \xi)\} = \{(x, \xi \mathbf{dx})\}. \end{aligned} \quad (16)$$

Proof. The calculation of C is well known, see e.g., [2, 4]. Since R_μ^* is the dual of R_μ , R_μ^* is an FIO associated to C^t by the standard calculus of FIO, e.g., [6, Theorem 4.2.1]. That $\Pi_L : C \rightarrow T^*(\Xi)$ is an injective immersion (The Bolker Assumption) is a straightforward calculation [4, 17].

One uses (14) to find the two preimages of $(x, \xi \mathbf{dx})$ under $\Pi_R : C \rightarrow T^*(\mathbb{R}^2)$ using the fact that $\xi = \|\xi\| \theta(\phi_0(\xi)) = -\|\xi\| \theta(\phi_1(\xi))$. Statement (16) follow from the observation that, if $A \subset T^*(\mathbb{R}^2)$, then $C \circ A = \Pi_L(\Pi_R^{-1}(A))$ (and if $B \subset T^*(\Xi)$, then $C^t \circ B = \Pi_R(\Pi_L^{-1}(B))$). \square

The next theorem provides a generalization to R_μ and arbitrary filter P of the artifact characterization in [2, 8].

Theorem 4.2. *Let $f \in \mathcal{E}'(\mathbb{R}^2)$ and let μ be a nowhere zero smooth 2π -periodic function on $\mathbb{R} \times \mathbb{R}^2$. Let P be a pseudodifferential operator on $\mathcal{E}'(\Xi)$*

$$\text{WF}(R_\mu^* P R_{\mu, (a,b)} f) \subset \text{WF}_{(a,b)}(f) \cup \mathcal{A}_{\{a,b\}}(f), \quad (17)$$

where

$$\text{WF}_{(a,b)}(f) = \text{WF}(f) \cap \mathcal{V}_{(a,b)}, \text{ and } \mathcal{V}_{(a,b)} = \{(x, \alpha \theta(\phi) dx) : \alpha \neq 0, \phi \in (a, b)\} \quad (18)$$

is the set of visible singularities and

$$\begin{aligned} \mathcal{A}_{(a,b)}(f) &= \{(x + t\theta^\perp(\phi), \alpha \theta(\phi) dx) : \\ &\quad \phi \in \{a, b\}, \alpha, t \neq 0, x \in L(\phi, s), (x, \alpha \theta(\phi)) \in \text{WF}(f)\} \end{aligned} \quad (19)$$

is the set of possible added artifacts.

Now, assume that μ is nowhere zero and the top order symbol of P is nowhere zero modulo lower order symbols on $\{(\phi, s, \alpha[t d\phi + ds]) : \phi \in (a, b), s \in \mathbb{R}, t \in \mathbb{R}, \alpha \neq 0\}$. Furthermore assume $b - a < \pi$. Then,

$$\text{WF}_{(a,b)}(f) \subset \text{WF}(R_\mu^* P R_{\mu,(a,b)} f). \quad (20)$$

The condition $b - a < \pi$ is reasonable in limited data problems because, if $b - a \geq \pi$, then every line is parameterized by $L(\phi, s)$ for $\phi \in (a, b)$.

Radon transforms detect singularities conormal to the set being integrated over (e.g., [4, 15, 18]), and the above theorem states this relation explicitly: only singularities $(x, \alpha\theta(\phi)) \in \text{WF}(f)$ with directions in the visible angular range, $\phi \in (a, b)$, can be reconstructed from limited angle data. Singularities of f outside of $[a, b]$ are smoothed. However, each singularity of f at $(x, \alpha\theta(\phi_0))$ for $\phi_0 = a, b$ generates a line of artifacts through x and normal to $\theta(\phi_0)$.

Proof. We use the paradigm presented in Section 3 to prove (17). By Proposition 4.1, we know that R_μ is a Fourier integral operator with the canonical relation given in (14). Thus, the step (a) of our paradigm is carried out.

For the step (b), we consider $A = (a, b) \times \mathbb{R}$ with $0 < a < b < \pi$ and compute

$$\text{WF}(\chi_{(a,b) \times \mathbb{R}}) = \{((\phi, s); \beta d\phi) : \phi \in \{a, b\}, \beta \neq 0, s \in \mathbb{R}\}. \quad (21)$$

Since $\text{WF}(\chi_{(a,b) \times \mathbb{R}})$ has no ds -component and at the same time the ds -component of $\text{WF}(R_\mu f)$ is always non-zero, we see that the non-cancellation condition (10) holds. This is step (c) of our paradigm. Hence, by Theorem 3.1, the product

$$R_{\mu,(a,b)} f = \chi_{(a,b)} \cdot R_\mu f \quad (22)$$

is well-defined and

$$\text{WF}(R_{\mu,(a,b)} f) \subset \mathcal{Q}((a, b) \times \mathbb{R}, C \circ \text{WF}(f)).$$

In the next step (cf. (d)), we calculate $\mathcal{Q}((a, b) \times \mathbb{R}, C \circ \text{WF}(f))$ using (12).

Since the condition $[\xi = 0 \text{ and } \eta = 0]$ is not allowed, the set $\mathcal{Q}((a, b) \times \mathbb{R}, C \circ \text{WF}(f))$ is a union of three sets:

$$\begin{aligned} \mathcal{Q}((a, b) \times \mathbb{R}, C \circ \text{WF}(f)) = & [(C \circ \text{WF}(f)) \cap \{((\phi, s), \eta) \in T^*(\Xi) : \phi \in (a, b)\}] \\ & \cup \text{WF}(\chi_{(a,b)}) \cup W_{\{a,b\}}(f), \end{aligned} \quad (23)$$

where the first set (in braces) corresponds to $\xi \neq 0, \eta = 0$, the second to $\xi = 0, \eta \neq 0$ and the third, $W_{\{a,b\}}(f)$, corresponds to $\xi \neq 0, \eta \neq 0$ in the definition of \mathcal{Q} . To calculate $W_{\{a,b\}}(f)$ note that covectors in $C \circ \text{WF}(f)$ are of the form $((\phi, s); \alpha(-\delta d\phi + ds))$ where there exists an $x \in L(\phi, s)$ with $(x, \alpha\theta(\phi)) \in \text{WF}(f)$ and where $\delta = x \cdot \theta^\perp(\phi)$. Also, $\eta \neq 0$ corresponds to covectors in $\text{WF}(\chi_{(a,b)})$, which are of the form $((\phi, s); \beta d\phi)$ where $\phi \in \{a, b\}$ and $\beta \neq 0$. Adding these vectors for the same base point, one sees that the covector $((\phi, s); (\beta - \alpha\delta) d\phi + \alpha ds)$ is in $W_{\{a,b\}}(f)$. Since β is arbitrary, one can write

$$\begin{aligned} W_{\{a,b\}}(f) = & \{((\phi, s); \nu d\phi + \alpha ds) : \\ & \nu \in \mathbb{R}, \alpha \neq 0, \phi \in \{a, b\}, \exists x \in L(\phi, s) : (x, \alpha\theta(\phi)) \in \text{WF}(f)\}. \end{aligned} \quad (24)$$

To accomplish the step (e) in our paradigm, we let P be a pseudodifferential operator. Then, by containment (13),

$$\text{WF}(R_\mu^* P R_{\mu,(a,b)} f) \subset C^t \circ \mathcal{Q}((a, b) \times \mathbb{R}, C \circ \text{WF}(f)).$$

We now compute $C^t \circ \mathcal{Q}((a, b) \times \mathbb{R}, C \circ \text{WF}(f))$. Using (23) and the composition rules, first observe that

$$C^t \circ \mathcal{Q}((a, b) \times \mathbb{R}, C \circ \text{WF}(f)) = C^t \circ [(C \circ \text{WF}(f)) \cap \{((\phi, s), \eta) \in T^*(\Xi) : \phi \in (a, b)\}] \cup C^t \circ \text{WF}(\chi_{(a,b)}) \cup C^t \circ W_{\{a,b\}}(f). \quad (25)$$

We examine the three terms of the equation (25) separately. First, we get

$$C^t \circ [(C \circ \text{WF}(f)) \cap \{((\phi, s), \eta) \in T^*(\Xi) : \phi \in (a, b)\}] = [(C^t \circ C) \circ \text{WF}(f)] \cap [C^t \circ \{((\phi, s), \eta) \in T^*(\Xi) : \phi \in (a, b)\}]. \quad (26)$$

It is not hard to see that $C^t \circ C = \Delta := \{(x, \xi dx; x, \xi dx) : (x, \xi dx) \in T^*\mathbb{R}^2\}$ and $\Delta \circ \text{WF}(f) = \text{WF}(f)$. Moreover,

$$C^t \circ \{((\phi, s), \eta) \in T^*(\Xi) : \phi \in (a, b)\} = \mathcal{V}_{(a,b)}.$$

Hence, the first set in (25) is equal to the set of visible singularities (18)

$$\text{WF}_{(a,b)}(f) = \text{WF}(f) \cap \mathcal{V}_{(a,b)}.$$

For the second set in (25) observe that $C^t \circ \text{WF}(\chi_{(a,b)}) = \emptyset$ since the ds -components of covectors in $\text{WF}(\chi_{(a,b)})$ is zero and the ds -components of covectors in C^t is always non-zero.

Finally, we consider the set $C^t \circ W_{\{a,b\}}(f)$. Let

$$\gamma = ((\phi, s); \nu d\phi + \alpha ds) \in W_{\{a,b\}}(f),$$

then $\nu \in \mathbb{R}$, $\alpha \neq 0$, $\phi \in \{a, b\}$, $s \in \mathbb{R}$, and there is a $x \in L(\phi, s)$ such that $(x, \alpha\theta(\phi)) \in \text{WF}(f)$. By the definition of composition, (6),

$$C^t \circ \{\gamma\} = \{(\tilde{x}, \alpha\theta(\phi) dx) : (\tilde{x}, \alpha\theta(\phi) dx; \gamma) \in C^t\}$$

where, by the definition of C^t , ((14) with the coordinates switched), $\tilde{x} \in L(\phi, s)$ so $s = \tilde{x} \cdot \theta(\phi)$. Let $t = -\nu/\alpha$. Since ν is arbitrary, t is arbitrary. Again by the definition of C^t , $t = -\nu/\alpha = \tilde{x} \cdot \theta^\perp(\phi)$, so the point $\tilde{x} = s\theta(\phi) + (-\nu/\alpha)\theta^\perp(\phi)$ is an arbitrary point in $L(\phi, s)$. Therefore, for any $\tilde{x} \in L(\phi, s)$, the covector $(\tilde{x}, \alpha\theta(\phi) dx) \in C^t \circ W_{\{a,b\}}(f)$. Thus, the third set in (25) is the set of possible added singularities given by (19).

Containment (20) is proven using Corollary 5.2 from the next section. Let $(x, \xi d\mathbf{x}) \in \text{WF}(f) \cap \mathcal{V}_{(a,b)}$. Then, one of the angles $\phi_0(\xi)$ or $\phi_1(\xi)$ (defined in Proposition 4.1) is in (a, b) and the other one is not since $b - a < \pi$. Without loss of generality, assume $\phi_0(\xi) \in (a, b)$.

Let φ be a cutoff function in ϕ that is supported in (a, b) and equal to one in a smaller neighborhood (a', b') of ϕ' . We will define \mathcal{K}_φ as the multiplication operator $\mathcal{K}_\varphi g(\phi, s) = \varphi(\phi)g(\phi, s)$.

Let $g_1 = P\mathcal{K}_\varphi R_\mu(f)$ and $g_2 = P[\chi_{(a,b)} - \varphi] R_\mu(f)$. By Corollary 5.3 part 2, the symbol of $R_\mu^* P\mathcal{K}_\varphi R_\mu$ is elliptic on $\mathcal{V}_{(a', b')}$ and so at $(x, \xi d\mathbf{x})$. Therefore, $(x, \xi d\mathbf{x}) \in \text{WF}(R_\mu^* g_1)$. We now show $(x, \xi d\mathbf{x}) \notin \text{WF}(R_\mu^* g_2)$. Because $\chi_{(a,b)} - \varphi$ is zero on (a', b') , $[\chi_{(a,b)} - \varphi] R_\mu f$ is zero on $(a', b') \times \mathbb{R}$. Therefore, $g_2 = P[\chi_{(a,b)} - \varphi] R_\mu(f)$ is smooth on $(a', b') \times \mathbb{R}$, and since $\phi_0(\xi) \in (a', b')$, $\lambda_0(x, \xi) \notin \text{WF}(g_2)$. Since $b - a < \pi$, $\phi_1(\xi) \notin (a, b)$, so g_2 is smooth near $\phi_1(\xi)$. This implies that $\lambda_1(x, \xi) \notin \text{WF}(g_2)$. Using the Hörmander-Sato Lemma 7, $\text{WF}(R_\mu^* g_2) \subset C^t \circ \text{WF}(g_2)$, so, by (16) the only two covectors, $\lambda_0(x, \xi)$ and $\lambda_1(x, \xi)$, that can contribute to wavefront of $R_\mu^* g_2$ at $(x, \xi d\mathbf{x})$ are not in $\text{WF}(g_2)$ so $(x, \xi d\mathbf{x}) \notin \text{WF}(R_\mu^* g_2)$.

Therefore, $(x, \xi d\mathbf{x}) \in \text{WF}(R_\mu^* g_1 + R_\mu^* g_2) = \text{WF}(\mathcal{L}_\varphi f)$, and this proves the final part of the theorem. \square

5 Reduction of artifacts

The singularity reduction method replaces the sharp cutoff $\chi_{(a,b)}$ by a smooth cutoff. Let φ be a smooth cutoff function supported in (a, b) and equal to one on a proper subinterval (a', b') , and replace $\chi_{(a,b)}$ by φ in the reconstruction operator. Then the artifact-reduced reconstruction operator is

$$\mathcal{L}_\varphi f = R_\mu^* P \mathcal{K}_\varphi R_\mu f \quad \text{where} \quad \mathcal{K}_\varphi g = \varphi g. \quad (27)$$

This method was analyzed for the lambda filter $P = -d^2/ds^2$ and the FBP filter $P = \sqrt{-d^2/ds^2}$ and with R_1 in [2] and with R_μ in [8, 9]. Our theorems provide generalization to arbitrary filters P , and they provide the symbol of \mathcal{L}_φ in general with proof.

Theorem 5.1. *Let μ be a smooth measure and let φ be a smooth function supported in (a, b) and equal to 1 on the proper subinterval (a', b') . Then*

$$\text{WF}(\mathcal{L}_\varphi(f)) \subset \text{WF}_{(a,b)}(f). \quad (28)$$

The top order symbol of \mathcal{L}_φ is

$$\begin{aligned} \sigma(\mathcal{L}_\varphi)(x, \xi \mathbf{d}x) = \frac{2\pi}{\|\xi\|} & \left[\varphi(\phi_0(\xi)) p(\lambda_0(x, \xi)) \mu^2(\phi_0(\xi), x) \right. \\ & \left. + \varphi(\phi_1(\xi)) p(\lambda_1(x, \xi)) \mu^2(\phi_1(\xi), x) \right] \end{aligned} \quad (29)$$

where P is a pseudodifferential operator on $\mathcal{E}'(\Xi)$ and the notation is given in (15).

If ν is a smooth weight and R_μ^* is replaced by R_ν^* , then the μ factor in (29) is replaced by $\nu\mu$.

Corollary 5.2. *Let φ be a nonnegative smooth function supported on (a, b) and equal to 1 on a subinterval (a', b') . Assume the symbol $\sigma(\mathcal{L}_\varphi)$ in (29) is nowhere zero modulo lower order symbols. Then,*

$$\text{WF}_{(a', b')}(f) \subset \text{WF}(\mathcal{L}_\varphi(f)). \quad (30)$$

This theorem shows that as long as P is well-chosen, most visible wavefront directions (those in $\text{WF}_{(a', b')}(f)$) are visible using the artifact reduced operator \mathcal{L}_φ and artifacts are not added since $\text{WF}(\mathcal{L}_\varphi(f))$ is contained in $\text{WF}_{(a,b)}(f)$. The proof follows from the ellipticity assumption in the corollary using, e.g., [20, Prop. 6.9].

Our next corollary provides specific cases in which the theorem can be applied.

Corollary 5.3. *Let φ be a nonnegative function supported in (a, b) and equal to 1 on the subinterval (a', b') . Let*

$$\mathcal{A} = \{(\phi, s, \alpha[t d\phi + ds]) : \phi \in (a', b'), s \in \mathbb{R}, t \in \mathbb{R}, \alpha \neq 0\}.$$

Then $\mathcal{L}_\varphi = R_\mu^ \mathcal{K}_\varphi P R_\mu$ is elliptic on $\mathcal{V}_{(a', b')}$ (therefore (30) holds) when either of the following conditions hold for μ and P :*

1. μ is real and nowhere zero and the top order symbol $\sigma(P) = p$ is real and nonzero on \mathcal{A} , or
2. $b - a < \pi$ and μ is nowhere zero and p is elliptic on \mathcal{A} .

Condition 1 holds, for example, if $P = -d^2/ds^2$, the filter in Lambda tomography, or $P = \sqrt{-d^2/ds^2}$, the filter in FBP because, in both cases, the symbol is positive on \mathcal{A} (e.g., $\sigma(\sqrt{-d^2/ds^2})(\phi, s, \beta d\phi + \alpha ds) = |\alpha|$), and our theorem can be applied to these operators.

If $b-a < \pi$ and $P = d/ds$, then condition 2 holds since the symbol of d/ds is nowhere zero on \mathcal{A} . Thus, \mathcal{L}_φ is elliptic on $\mathcal{V}_{(a', b')}$. However, if $b-a > \pi$, ellipticity of P is not sufficient for ellipticity of \mathcal{L}_φ . For example, consider the full data problem for the classical transform R_1 , then $\sigma(P)(\phi, s, \beta d\phi + \alpha ds) = \alpha$ changes sign on \mathcal{A} and the operator $R_1^*(d/ds R_1) = 0$ by symmetry.

Proof of Theorem 5.1. We use the notation, conventions, and symbol calculation in [17, Theorem 3.1]. Recall that $\Pi_R : C \rightarrow T^*(\mathbb{R}^2)$ and $\Pi_L : C \rightarrow T^*(\Xi)$ are the natural projections. Equation (14) in [17] and the discussion below it give the symbol of R_μ as the half density

$$\sigma(R_\mu) = \frac{(2\pi)^{1/2} \mu(\phi, x) d\phi dx \sqrt{dw d\eta}}{\sqrt{d\phi ds dx} \Pi_R^*(|\sigma_{\mathbb{R}^2}|)} \quad (31)$$

where $|\sigma_{\mathbb{R}^2}|$ is the density from the canonical symplectic form on $T^*(\mathbb{R}^2)$ and $\Pi_R^*(|\sigma_{\mathbb{R}^2}|)$ is its pull back to C . Also, $Z = \{(\phi, x \cdot \theta(\phi), x) : \phi \in [0, 2\pi), x \in \mathbb{R}^2\}$ is the set in $\Xi \times \mathbb{R}^2$ over which the Schwartz kernel of R_μ integrates, and $z = (\phi, x \cdot \theta(\phi), x)$ and $w = x \cdot \theta(\phi) - s$ give coordinates on $\Xi \times \mathbb{R}^2$. Then, the *measure* on Z associated to R_μ is $\mu(\phi, x) d\phi dx$ (see equation (16) in [17]). Finally η is the fiber coordinate in the conormal bundle of Z . An analogous argument shows that the symbol of R_μ^* is given by

$$\sigma(R_\mu^*) = \frac{(2\pi)^{1/2} \mu(\phi, x) d\phi dx \sqrt{dw d\eta}}{\sqrt{d\phi ds dx} \Pi_L^*(|\sigma_\Xi|)}. \quad (32)$$

The pseudodifferential operator PK_φ has symbol $\varphi(\phi)p(\phi, s, \gamma)$ (where $\gamma \in T_{(\phi, s)}^*(\Xi)$) so $PK_\varphi R_\mu$ is a standard smooth FIO and its top order symbol is

$$\sigma(PK_\varphi R_\mu) = \frac{(2\pi)^{1/2} p(\phi, s, \gamma) \varphi(\phi) \mu(\phi, x) d\phi dx \sqrt{dw d\eta}}{\sqrt{d\phi ds dx} \Pi_R^*(|\sigma_{\mathbb{R}^2}|)}$$

when evaluated at covectors on C .

Let $(x, \xi \mathbf{dx}) \in T^*(\mathbb{R}^2) \setminus \mathbf{0}$. To calculate the symbol of the composition of R_μ^* with $PK_\varphi R_\mu$ one uses the note at the top of p. 338 of [17]: since the projection $\Pi_R : C \rightarrow T^*(\mathbb{R}^2) \setminus \mathbf{0}$ is two-to-one, the symbol of $R_\mu^* PK_\varphi R_\mu$ at $(x, \xi \mathbf{dx}) \in T^*(\mathbb{R}^2)$ is the sum of the product $\sigma(R_\mu^*) \cdot \sigma(PK_\varphi R_\mu)$ at the two preimages. By Proposition 4.1, those preimages, given by $\Pi_R^{-1}(x, \xi \mathbf{dx})$, are the two covectors

$$(\lambda_0(x, \xi); x, \xi \mathbf{dx}) \text{ and } (\lambda_1(x, \xi); x, \xi \mathbf{dx}).$$

Under the conventions of [17], the symbol of $R_\mu^* PK_\varphi R_\mu$ at $(x, \xi \mathbf{dx})$ is the sum

$$\begin{aligned} \sigma(R_\mu^* PK_\varphi R_\mu)(x, \xi \mathbf{dx}) = & \left\{ \frac{2\pi (d\phi dx)^2 dw d\eta}{d\phi ds dx \Pi_R^*(|\sigma_{\mathbb{R}^2}|) \Pi_L^*(|\sigma_\Xi|)} \right\} \\ & \times [\varphi(\phi_0(\xi)) \mu^2(\phi_0(\xi), x) p(\lambda_0(x, \xi)) \\ & + \varphi(\phi_1(\xi)) \mu^2(\phi_1(\xi), x) p(\lambda_1(x, \xi))] \end{aligned} \quad (33)$$

Now, [17, Lemma 3.2] shows, for the Radon line transform, that the term on the top right in braces in (33) can be simplified to equal to $2\pi/\|\xi\|$. Putting this into (33) proves the symbol calculation (29). \square

Proof of Corollary 5.3. In each case, we will show that $\sigma(\mathcal{L}_\varphi)$ is elliptic on $\mathcal{V}_{(a',b')}$. Let $(x, \xi \mathbf{d}\mathbf{x}) \in \mathcal{V}_{(a',b')}$, then either $\phi_0(\xi)$ or $\phi_1(\xi)$ or both are in (a', b') . Without loss of generality, we assume $\phi_0(\xi) \in (a', b')$. Therefore, $\varphi(\phi_0(\xi)) = 1$.

In case 1 we assume μ is real and nowhere zero and the top order symbol of P , $\sigma(P) = p$, is real and nowhere zero on \mathcal{A} . Therefore p is either always positive or always negative on \mathcal{A} . Since $\varphi = 1$ on (a', b') and $\mu^2 > 0$, at least the first term in brackets in (29) (the one containing ϕ_0) is nonzero. The second term (containing $\phi_1(\xi)$) either has the same sign as this term (since the sign of p does not change) or is zero (if $\phi_1(\xi) \notin \text{supp}(\varphi)$). Therefore the sum is nonzero and so the symbol of \mathcal{L}_φ is elliptic on $\mathcal{V}_{(a',b')}$.

In case (2), since $b - a < \pi$ and $\phi_0(\xi) \in (a', b')$, $\phi_1(\xi) \notin (a', b')$. Therefore, only one term in brackets in (29) is nonzero. Therefore, the symbol is elliptic on $\mathcal{V}_{(a',b')}$. \square

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